INVESTMENT DECISIONS WITH STOCHASTIC AND FUZZY INFORMATION

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Investment decisions of a firm are important decisions as they determine its earning generating capacity and thereby, ultimately its long term solvency. Such decisions involve allocation of a firm's resources among various fixed assets with an objective of maximising shareholders' value. These decisions are undertaken in a highly complex and uncertain environment that makes them extremely difficult and risky. Not only that, the information available is of heterogeneous nature. Since such decisions have futuristic dimensions, it is always plausible to assume that some available information may be stochastic and some may be fuzzy in nature[1]. At present, decision-makers usually ignore fuzzy information and make decisions on the basis of stochastic information or make some unrealistic assumptions to convert fuzzy information to stochastic information so as to use the existing tools of Probability Theory. Thus, in either case, it is difficult to expect sound investment decisions. Therefore, decision-makers should be equipped with a model, which allows them to incorporate stochastic as well as fuzzy information and process them in an integrated manner for more meaningful investment decisions. An attempt is made here to suggest a mathematical programming model - Possibilistic-Chance-Constrained Model, that can incorporate both stochastic as well as fuzzy information.

What follows is planned as thus: Section-I introduces the model; Section-II discusses how to convert possibilistic constraint and possibilistic - chance constraint into crisp constrainst; in Section-III, a numerical example is taken; and finally, conclusion.

SECTION-I

The model, Possibilistic-Chance-Constrained Model, suggested for investment decisions

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1 For precise distinction between stochastic information and fuzzy information, see Gupta, C.P. (1992).

is a mathematical programming model in that the relevant objective function is to be optimised subject to a number of constraints about which information available may be crisp certain, stochastic and/or fuzzy. More precisely, one can make use of it when

(i) the firm's objective is one and crisp; and,

(ii) information available about constraints is of crisp certain, stochastic and possibilistic[1] nature.

It may me stated as thus,

Optimize CX **[FSMP]**
Subject to

$$G(X)$$
$$X \geq 0$$

And, G(X) may be of any of the following:

$$\text{Prob.}\left(\sum_{i=1}^{n} a_{ij} x_i \leq \hat{b}_j\right) \geq p_j = 1,2,\ldots,m \text{ (Crisp Constraint)}$$

$$\left(\sum_{i=1}^{n} a_{ij}\ x\ i \leq b_j\right) \geq p_j = 1,2,\ldots,m \text{ (Chance Constraint)}$$

$$\text{Poss}\left(\widetilde{\sum_{i=1}^{n}} \tilde{a}_{ij} x_i \leq \tilde{b}_j\right) \geq \alpha_j\ j = 1,2,\ldots,m \text{ (Possibilistic Constraint)}$$

$$\text{Poss}\left(\text{Prob}\left(\widetilde{\sum_{i=1}^{n}} \tilde{a}_{ij}\ x_i \leq \hat{b}_j\right) \geq p_j\right) \geq \alpha_j\ j=1,2,\ldots m$$

(Possibilistic-Chance-Constraint)

where

$\hat{b}_j$ follows an independent and normal probability distribution with parameters - $(m_{\hat{b}j}, s_{\hat{b}j})$; m = mean and s = standard deviation;

p_i = the degree of probability that j^{th} constraint is not violated;

$\tilde{a}_{ij}$ and $\tilde{b}_j$ are fuzzy numbers.[2]

α_j is the degree of possibility by which j^{th} constraint should be satisfied.

Σ is fuzzy summation.

One can have the following interpretations of the above constraints:

1. *Crisp Constraint:* Such a constraint should be satisfied in a strict sense and they are hard constraints.
2. *Stochastic Constraint:* Such a constraint should be satisfied with at least p probability.
3. *Possibilistic Constraint:* Such a constraint should be satisfied with at least α possibility and they are soft constraints.
4. *Possibilistic-Stochastic Constraint:* Such a constraint should be satisfied with at least α possibility of having probability of not violating a constraint is at least p.

Such a model will be finally converted into a crisp Linear Programming model. Crisp constraints do not require any transformation and they come into the final model as such. Chance constraints can be converted into crisp constraints following Byrne et.el. [1971]. However, the conversion of possibilistic constraints and possibilistic-chance constraints into crisp constraints is discussed below.

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1 On every fuzzy set, one can easily define a possibility function in a natural way and thereby, a fuzzy information can be transformed into possibilistic one. Since, such a natural transformation is always available, here we have used the terms-fuzzy information and possibilistic information - inerchangeably. For details, see Zadeh [1978].

2 For simplicity, they are assumed to be symmetrical triangular fuzzy numbers (STFN)

SECTION-II

Section-II discusses conversion of possibilistic constraint and possibilistic - chance constraint. First, we take how to convert a possibilistic constraint into a crisp constraint. For that, we state the following two important results:

Result #1: Let $\check{n}$ be a STFM, C be a crisp ordinary real number and $\otimes$ be fuzzy multiplication. Then, C $\otimes$ $\check{n}$ will also be a STFM.

Result #2: Let $\check{n}$ and $\check{m}$ be two STFMs and $\oplus$ be fuzzy addition. Then, $\check{n} \oplus \check{m}$ will also be a STFM.

If we assume that x_j's are to be crisp then from the above stated results, we get

$$\textcircled{\Sigma}_{i=1}^{n} \tilde{a}_{ij}\, x_j \text{ as a STFM.}$$

Further, to have a logical comparison between two fuzzy numbers, i.e., $\textcircled{\Sigma}_{i=1}^{n} \tilde{a}_{ij}\, x_j$ and b_j, we follow two definitions due to Dubois and Prade (1980):

Definition #1: Let $\tilde{M}$ and $\tilde{N}$ be two fuzzy numbers. The degree of possibility of $\tilde{M} \leq \tilde{N}$ is defined as:

$$\text{Poss}(\tilde{M} \leq \tilde{N}) = \sup_{x,y:x \leq y} \min(\mu_{\tilde{M}}(x)\ \mu_{\tilde{N}}(y))$$

Since $\tilde{M}$ and $\tilde{N}$ are convex fuzzy sets, it can **be seen** from Figure-1 that

Poss $(\tilde{M} \leq \tilde{N}) = 1$ iff $m \leq n$ where $\mu_{\tilde{M}}(m) = \mu_{\tilde{N}}(n) = 1$

and

$$\text{Poss}(\tilde{M} \geq \tilde{N}) = \text{hgt}(\tilde{M} \cap \tilde{N}) = \mu_{\tilde{M}}(d) = \mu_{\tilde{N}}(d)$$

Definition #2: Let M and N be two fuzzy numbers. The degree of possibility of $\tilde{M}=\tilde{N}$ is defined as

$$\text{Poss}(\tilde{M}=\tilde{N}) = \text{Min.}(\text{Poss}(\tilde{M} \leq \tilde{N}), \text{Poss}(\tilde{M} \geq \tilde{N}))$$

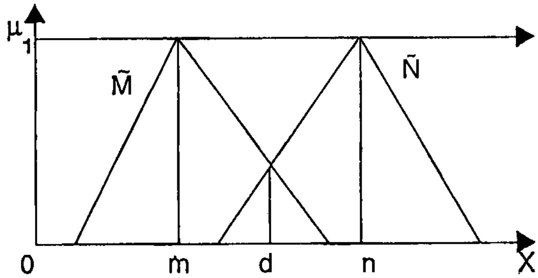

Figure-1

To see the implications of (Definition #1) and (Definition #2) for STFN and for (FSMP), we consider two STFNs, as shown in Figure-2. From (Definition #1), we obtain Poss $(\tilde{B} \leq \tilde{A})$, $\mu_{\tilde{A}}(H_0) = \mu_{\tilde{B}}(h_0) = \alpha$.

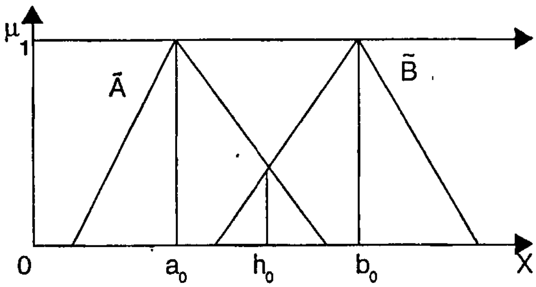

Figure-2

We note that h_0 is obtained at a point where $\mu_{\tilde{A}}(.)$ is decreasing and $\mu_{\tilde{B}}(.)$ is increasing. Thus, the intersection point gives the following equality:

$$a_0 + (1 - \alpha)\, d_a = b_0 - (1 - \alpha)\, d_b \qquad (1)$$

where

d_a = spread of $\tilde{A}$

and, d_b = spread of $\tilde{B}$

But, if we wish that Poss $(\tilde{B} \leq \tilde{A}) \geq \alpha$ then, as indicated by Figure-2, we must have

$$a_0 + (1 - \alpha)\, d_a \geq b_0 - (1 - \alpha)\, d_b \quad (2)$$

Thus, from (2), we can conclude that

(a) Poss $(\tilde{A} \leq \tilde{B}) \geq \alpha \Leftrightarrow a_0 - (1-\alpha)d_a \leq b_0 + (1-\alpha)d_b$ (3.1)

(b) Poss $(\tilde{A} \geq \tilde{B}) \geq \alpha \Leftrightarrow a_0 + (1-\alpha)d_a \geq b_0 - (1-_x)d_b$ (3.2)

Also, from (Definition #2), we obtain

Poss $(\tilde{A} \approx \tilde{B})$ = Min (Poss$(\tilde{A} \leq \tilde{B})$, Poss $(\tilde{A} \geq \tilde{B})$)

Thus,

Poss $(\tilde{A} \approx \tilde{B}) \geq \alpha \Leftrightarrow$ Poss $(\tilde{A} \leq \tilde{B}) \geq \alpha$ and Poss $(\tilde{A} \geq \tilde{B}) \geq \alpha$

Therefore, about Poss $(\tilde{A} = \tilde{B}) \geq \alpha$, we conclude that

Poss$(\tilde{A} = \tilde{B}) \geq \alpha \Leftrightarrow$

(a) Poss $(\tilde{A} \leq \tilde{B}) \geq \alpha \Leftrightarrow a_0 - (1-\alpha)d_a \leq b_0 + (1-\alpha)d_b$
(b) Poss $(\tilde{A} \geq \tilde{B}) \geq \alpha \Leftrightarrow a_0 + (1-\alpha)d_a \geq b_0 - (1-\alpha)d_b$ (4)

Thus, using results of (3.1), (3.2) and (4), one can convert a possibilistic constraint into a crisp constraint.

Now, we see how to convert a possibilistic-chance constraint into a crisp constraint. Consider first chance part of possibilistic-chance constraint. Since $\hat{b}_j$ is a random variable (**r.v.**) in j^{th} constraint; it can be satisfied only in probabilistic sense. Hence, the firm has the following chance constraint:

$$\text{Prob}\left(\sum_{i=1}^{n} \tilde{a}_{ij}\, x_i \leq \hat{b}_j\right) \geq p_j \quad (5)$$

But, as $\tilde{a}_{ij}$s are fuzzy numbers defining possibility distributions, we assume that the management further restricts (5) by a possibilistic constraint. Thus, the new constraint would mean '*the possibility that the probability of not violating a constraint is p_j or more must be at least α*'.

$$\text{Poss}\left(\text{Prob}\left(\sum_{i=1}^{n} \tilde{a}_{ij} x_i \leq \hat{b}_j\right) \geq p_j\right) > \alpha_j\; j = 1,2,\ldots,m$$

Again, for simplicity, we assume that a_{ij}s are STFNs. Since a_{ij}s are STFNs, they induce a possibility distribution for

$$\text{Prob}\left(\sum_{i=1}^{n} \tilde{a}_{ij} x_i \leq \hat{b}_j\right) \geq p_j$$

Now, let's see below how it is induced.

Consider a random variable (**r.v.**) x having a normal probability distribution with parameters (m,s). Then,

$$\text{Prob.}(x \leq A) = \text{Prob.}(x \leq (A-m)/s) = F(z) \quad (6)$$

where z is a standard normal variate with parameters (1,0). In (6), if A is a fuzzy number, then z and F(z) will also be fuzzy numbers, $\tilde{z}$ and $F(\tilde{z})$ respectively. From Result (#1) and Result (#2), we can conclude that if A is a STFN and, m and s are crisp numbers, then $\tilde{z} = ((\tilde{A} - m)/s)$ will also be a STFN. Further, the membership of $\tilde{z}$ and $F(\tilde{z})$ can be determined by using Zadeh's Extension Principle as thus:

$$\mu_{\tilde{Z}}(z) = \mu_{\tilde{A}}(a \mid z = ((a-m)/s); \text{ and} \quad (7)$$

$$\mu_{F(z)}(F(z)) = \mu(\tilde{z} \mid F(z) \text{ holds})$$

$$= \mu_{\tilde{A}}(a \mid F((a-m/s) \text{ holds}) \quad (8)$$

Following Wierzchen (1988), we assume:

(i) $P(x \leq \tilde{A}) = 1 - P(x \geq \tilde{A})$ and; (9)

(ii) $P(x = \tilde{A}) = 0$ **(10)**

(9) and (10) are reasonable assumptions when x follows a normal probability distribution in the sense that their crisp analogues are always true, i.e.

(i) $P(x \leq A) = 1 - P(x \geq A)$ and;

(ii) $P(x = A) = 0$.

The assumptions (9) and (10) have following two important implications for the Model:

(i) a constraint of the nature

$$\text{Poss}\left(\text{Prob}\left(\sum_{i=1}^{n} \tilde{a}_i x_i \leq \hat{b}_j\right) \geq p_j \geq \alpha_j\right.$$

can be modeled on same lines are

$$\text{Poss}\left(\text{Prob}\left(\sum_{i=1}^{n} \tilde{a}_{ij} x_j \geq \hat{b}_j\right) = p_j\right) \geq \alpha_j$$

and;

(ii) in our model, no constraint will be of the following nature:

$$\text{Poss}\left(\text{Prob}\left(\sum_{i=1}^{n} \tilde{a}_{ij} x_j = \hat{b}_j\right) \geq p_j\right) \geq \alpha_j$$

Here, first we see how a constraint

$$\text{Poss}\left(\text{Prob}\left(\sum_{i=1}^{n} \tilde{a}_{ij} x_j \geq \hat{b}_j\right) \geq p_j\right) \geq \alpha_j$$

can be converted into equivalent crisp **constraint** using the assumptions made above. For that, we consider

$$\text{Poss}(F(\tilde{z}) \geq p) \geq \alpha \quad (11)$$

where $F(\tilde{z})$ is a STFN representing Prob. ($\tilde{z} = \tilde{A}$-m/)s) in which $\tilde{A}$ is a STFN, m and s are respectively mean and standard deviation of a normal probability distribution and; p and α are ordinary crisp numbers. Then, from Result (#1), we obtain that

$$\text{Poss}(F(\tilde{z} \geq p) = \sup_{F(z) \geq p} \mu_{F(\tilde{z})}(F(z))$$

$$= \sup_{z \geq (F^{-1}(p))} \mu_{\tilde{z}}(z) \quad (12)$$

$$= \sup_{((a-m)/s) \geq F^{-1}(p)} \mu_{\tilde{A}}(a)$$

Since p is a crisp number in (11), using (12) and (3.2), we obtain for Poss $(F(\tilde{z}) \geq p) \geq \alpha$ the following crisp equivalent:

$$\{ ((a_0 + (1+\alpha)d_a) - m)/s\} \geq F^{-1}(p)$$

$$\Rightarrow (a_0 + (1-\alpha)d_a) \geq m + sF^{-1}(p) \quad \mathbf{(13)}$$

Likewise and using the assumption that $P(x \leq \tilde{A}) = 1 - P(x \geq \tilde{A})$, we can obtain for Poss $((1-F(z)) \geq p) \geq \alpha$ the following crisp equivalent $\{((a_0 + (1-\alpha)d_a) - m)/s\} > F^{-1}(1-p)$

$$\Rightarrow (a_0 - (1-\alpha)d_a) \geq m + sF^{-1}(1-p) \quad \mathbf{(14)}$$

Using the results of this section, we can convert our model (FSMP) into the following crisp L.P. Model:

Optimize CX

Subject to

$$\sum_{j=1}^{n} ((a_0)_{ij} - (1-\alpha_j)(d_a) x_{ij} \leq (b_o)_i + (1-\alpha)\, d_{bj}$$

[FSMP#1]

$$\sum_{j=1}^{n} ((a_0)_{ij} + (1-\alpha_i)(d_a)_{ij})\, x_{ij} \geq m + sF^{-1}(p)_i$$

$$X \geq 0$$

where

$(a_0)_{ij}$ = mean value of $\tilde{a}_{ij}$;

$(d_a)_{ij}$ = spread of $\tilde{b}_{ij}$;

(b_0) = mean value of $\tilde{b}_j$;

$(d_0)_i$ = spread of $\tilde{b}_j$;

m = mean of a normally distributed r.v. $\hat{b}$

s = standard deviation of $\hat{b}$

Now, one can solve **[FSMP#1]** using simplex method. Its application is illustrated below.

Section-III

NUMERICAL ILLUSTRATION

To illustrate the application of the model, we consider the problem given in Table-I.

	PROJECTS									bi
	1	2	3	4	5	6	7	8	9	
NPV (Rs)	14	17	17	15	40	12	40	10	12	
Environment Pollution (Points) [EP]	1.2	6.3	2.7	2.2	8.8	2.0	5.7	5.9	3.2	12 points
Sales [Period-I] (Rs [SP-I]	14	30	13	11	53	10	32	21	12	Rs 60
Sales Period-II] (Rs [SP-I])	15	42	16	12	52	14	34	28	21	Rs 60
Casoutflow (Periiod-I] (Rs) [SP-I]	12	54	6	6	30	6	48	36	18	Rs 60
Cashoutflow [Period-II(Rs)[SP-II]	3	7	6	2	35	6	4	3	3	Rs 40
Technical Expertise (days) [TE]	20	80	18	14	88	15	74	60	28	110 days
Net Working Capital (Rs) [NWC]	5	11	7	4	3	5	12	9	5	Rs 25

Table-I *(Rs. are in Crores)*

The exact nature of the constraints is stated in the Table-II and the fuzzy coefficients of possibilistic constraints and possibilistc - chance constraints are given in theTable-III.

OBJECTIVE			MAX.NPV(X)
CONSTRAINTS	1. Environment Pollution	:	$\text{Poss}(EP(X)=12\pm0.2)\geq0.8$
	2. Sales Period I	:	$\text{Poss}(SP\text{-}I(X)\geq60\pm10)\geq0.9$
	3. Sales Period II	:	$\text{Poss}(SP\text{-}II)(X)\ \geq\ 69+15)\geq0.7$
	4. Cashoutflows Period I	:	$CF\text{-}I(X)\ \leq\ 60$
	5. Cashoutflows Period II	:	$CF\text{-}II(X)\ \leq\ 40$
	6. Technical Expertise	:	$\text{Poss}(\text{Prob}(TE(X)\ \leq\ \hat{b}_7)\geq0.99)\geq0.5$
	7. Net Working Capital	:	$\text{Poss}(\text{Prob}\ (NWC(X) < \hat{b}_8)>0.95)\geq0.6$

Table-II

Constraints	$\tilde{a}_{1j}$	$\tilde{a}_{2j}$	$\tilde{a}_{3j}$	$\tilde{a}_{4j}$	a_{5j}	$\tilde{a}_{6j}$	$\tilde{a}_{7j}$	$\tilde{a}_{8j}$	$\tilde{a}_{9j}$
EP(X)	1.2±0.2	6.3±0.3	2.7±0.3	2.2±0.2	8.8±0.4	2.0±0.1	5.7±0.3	5.9±0.1	3.2±0.2
SP-I(X)	14±2	0±3	13±3	11±2	53±3	10±2	32±3	21±4	12±2
SP-II(X)	15±2	42±4	16±2	12±2	52±2	14±3	34±2	28±3	21±2
TE(X)	20±2	80±5	18±2	14±2	88±6	15±3	74±4	60±3	28±2
NWC(X)	5±1	11±1	7±2	4±0.5	3±0.5	5±1	12±2	9±1	5±0.5

Table-III

Using the results of Section-II, we obtain the equivalent crisp LP model of the above investment problem as thus:

MAX. $NPV(X) = 14x1+17x2 + 17x3 + 15x4 + 40x5 + 12x6 + 14x7 + 10x8 + 12x9$
suibject to

EP(X)
$1.16x1 + 6.24x2 + 2.64x3 + 2.16x4 + 8.72x5 + 1.98x6 + 5.64x7 + 5.88x8 + 3.16x9 \leq 12.04$

$1.24x1 + 6.36x2 + 2.76x3 + 2.24x4 + 8.88x5 + 2.02x6 + 5.76x7 + 5.92x8 + 3.24x9 \geq 11.96$

SP-I(X): $14.2x1 + 30.3x2 + 13.3x3 + 11.2x4 + 53.5x5 + 10.2x6 + 32.3x7 + 21.4x8 + 12.2x9 \geq 59$

SP-II(X): $15.9x1 + 43.2x2 + 16.6x3 + 12.6x4 + 53.2x5 + 14.9x6 + 34.6x7 + 28.9x8 + 21.6\ 9 \geq 64.5$

CF-I(X): $12x1 + 54x2 + 6x3 + 6x4 + 30x5 + 6x6 + 48x7 + 36x8 + 18x9 \leq 60$

CF-II(X): $3x1 + 7x2 + 6x3 + 2x4 + 35x5 + 6x6 + 4x7 + 3x8 + 3x9 \leq 40$

TE(X): $19x1 + 77.5x2 + 17x3 + 13x4 + 85x5 + 13.5x6 + 72x7 + 58.5x8 + 27x9 \leq 125$

NWC(X): $4.6x1 + 10.6x2 + 6.2x3 + 3.8x4 + 2.8x5 + 4.6x6 + 11.2x7 + 8.6x8 + 4.9x9 \leq 27$

$x1, x2, \ldots, x9 \geq 0$

The above model is solved and the results are summarized in Table-IV.

Projects	Values	Goal		Achievement
1	2.9363	NPV(X)		Rs. 98.29
2	0	Constraints	Value	Possibility of their satisfaction
3	0			
4	3.4511	EP(X)	Points 12.04	0.80
5	0.1353	SP-I(X)	Rs. 87.59	1.00
6	0	SP-II(X)	Rs. 97.37	1.00
7	0	CF-I(X)	Rs. 60.00	-
8	0	CF-II(X)	Rs. 20.45	-
9	0	TE(X)	Days 112.15	0.785
		NWC(X)	Rs. 27.00	0.600

Table-IV

CONCLUSION

Thus, we find that the model suggested is capable of integrating all kinds of information - certain, probabilistic and fuzzy in an investment decision. Since the different kinds of uncertainties investors are facing in present world, we believe that such an attempt would allow them to process heterogeneous information available in one model and that to in an integrated way.

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